# SEM 1, 2023-24: ALGEBRAIC TOPOLOGY
# END-SEMESTRAL EXAMINATION

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Max score: 50 marks. Time: 3 hours.
You may use any theorem proved in class in your solution, but please quote it fully and clearly.

(1) Compute the fundamental group and the singular homology groups of the surface $\Sigma_g$ for $g \geq 0$. Recall that $\Sigma_g$ is obtained from a sphere by attaching $g$ copies of the torus $\mathbb{S}^1 \times \mathbb{S}^1$. (8 marks)

(2) For a finite CW-complex $X$ and a $d$-sheeted covering $\tilde{X} \to X$, show that the Euler characteristic $\chi(\tilde{X}) = d \cdot \chi(X)$. (8 marks)

(3) Prove that if $f : \mathbb{S}^n \to \mathbb{S}^n$ has no fixed points then deg $f = (-1)^{n+1}$. Use this to prove that for $n = 2m$, there is some point $x \in \mathbb{S}^{2m}$ with either $f(x) = x$ or $f(x) = -x$. Deduce that every map $\mathbb{R}P^{2m} \to \mathbb{R}P^{2m}$ has a fixed point. (8 marks)

(4) Fix $n \geq 1$. Let $f : \mathbb{S}^n \to \mathbb{S}^n$ be a deck transformation of a certain covering space map $\mathbb{S}^n \to X$. What can you say about the degree of $f$? (8 marks)

(5) The Jordan curve theorem states that for any continuous, injective map $g : \mathbb{S}^{m-1} \to \mathbb{R}^m$, $\mathbb{R}^m \setminus g(\mathbb{S}^{m-1})$ has exactly two path-connected components. Using this or otherwise prove that there is no continuous injective map $\mathbb{S}^n \to \mathbb{R}^n$. (8 marks)

(6) Show that $\mathbb{S}^1 \times \mathbb{S}^1$ and $\mathbb{S}^1 \vee \mathbb{S}^1 \vee \mathbb{S}^2$ have isomorphic homology groups in all dimensions, but their universal covering spaces do not. (8 marks)

(7) Consider the properly discontinuous action of $\mathbb{Z}$ on $X = \mathbb{R}^2 \setminus 0$ given by $(x, y) \mapsto (2^n x, y/2^n)$ for $n \in \mathbb{Z}$. Compute the fundamental group of the quotient $Y = X/\mathbb{Z}$. Hint: Note that the covering $X \to Y$ is regular. (8 marks)

(8) If $m < n$, the embedding of $\mathbb{R}^{m+1}$ in $\mathbb{R}^{n+1}$ induces an embedding of $\mathbb{R}P^m$ in $\mathbb{R}P^n$. Compute the relative homology $H_k(\mathbb{R}P^n, \mathbb{R}P^m)$ for all $k, n, m$ with $n > m$. Note that your answer may depend on the parity of $m$ and $n$. (10 marks)

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*Date*: 22-11-2023.